Problem 3.43

Extended uncertainty principle.[43] The generalized uncertainty principle (Equation 3.62) states that

$$\sigma_A^2 \sigma_B^2 \geq \frac{1}{4}\langle C\rangle^2,$$

where $\hat{C} \equiv -i\left[\hat{A}, \hat{B}\right]$.

(a) Show that it can be strengthened to read

$$\sigma_A^2 \sigma_B^2 \geq \frac{1}{4}\left(\langle C\rangle^2 + \langle D\rangle^2\right), \tag{3.115}$$

where $\hat{D} \equiv \hat{A}\hat{B} + \hat{B}\hat{A} - 2\langle A\rangle\langle B\rangle$. *Hint:* Keep the $\operatorname{Re}(z)$ term in Equation 3.60.

(b) Check Equation 3.115 for the case $B = A$ (the standard uncertainty principle is trivial, in this case, since $\hat{C} = 0$; unfortunately, the extended uncertainty principle doesn't help much either).

Solution

Part (a)

Consider two observables, $A(x, p)$ and $B(x, p)$, represented by hermitian operators, $\hat{A}$ and $\hat{B}$, with real expectation values, $\langle A\rangle$ and $\langle B\rangle$. The respective variances of these observables are defined by (see Equation 1.10 and Equation 1.11 on page 11)

$$\sigma_A^2 = \left\langle(\Delta A)^2\right\rangle = \left\langle(A - \langle A\rangle)^2\right\rangle$$

$$\sigma_B^2 = \left\langle(\Delta B)^2\right\rangle = \left\langle(B - \langle B\rangle)^2\right\rangle.$$

Take their product.

$$\begin{aligned}
\sigma_A^2\sigma_B^2 &= \left\langle(A - \langle A\rangle)^2\right\rangle\left\langle(B - \langle B\rangle)^2\right\rangle \\
&= \left\langle\Psi\,\middle|\,\left(\hat{A} - \langle A\rangle\right)^2\,\middle|\,\Psi\right\rangle\left\langle\Psi\,\middle|\,\left(\hat{B} - \langle B\rangle\right)^2\,\middle|\,\Psi\right\rangle \\
&= \left[\int_{-\infty}^{\infty}\Psi^*(x,t)\left(\hat{A} - \langle A\rangle\right)^2\Psi(x,t)\,dx\right]\left[\int_{-\infty}^{\infty}\Psi^*(x,t)\left(\hat{B} - \langle B\rangle\right)^2\Psi(x,t)\,dx\right] \\
&= \left[\int_{-\infty}^{\infty}\Psi^*(x,t)\left(\hat{A} - \langle A\rangle\right)\left(\hat{A} - \langle A\rangle\right)\Psi(x,t)\,dx\right]\left[\int_{-\infty}^{\infty}\Psi^*(x,t)\left(\hat{B} - \langle B\rangle\right)\left(\hat{B} - \langle B\rangle\right)\Psi(x,t)\,dx\right] \\
&= \left\{\int_{-\infty}^{\infty}\left[\left(\hat{A} - \langle A\rangle\right)^\dagger\Psi(x,t)\right]^*\left[\left(\hat{A} - \langle A\rangle\right)\Psi(x,t)\right]dx\right\} \\
&\qquad\times\left\{\int_{-\infty}^{\infty}\left[\left(\hat{B} - \langle B\rangle\right)^\dagger\Psi(x,t)\right]^*\left[\left(\hat{B} - \langle B\rangle\right)\Psi(x,t)\right]dx\right\}
\end{aligned}$$

[43] For interesting commentary and references, see R. R. Puri, *Phys. Rev. A* **49**, 2178 (1994).

Since the operators are hermitian, $\hat{A}^\dagger = \hat{A}$ and $\hat{B}^\dagger = \hat{B}$. And since the expectation values are real, $\langle A\rangle^* = \langle A\rangle$ and $\langle B\rangle^* = \langle B\rangle$.

$$\begin{aligned}
\sigma_A^2\sigma_B^2 &= \left\{\int_{-\infty}^{\infty}\left[\left(\hat{A}^\dagger - \langle A\rangle^*\right)\Psi(x,t)\right]^*\left[\left(\hat{A} - \langle A\rangle\right)\Psi(x,t)\right]dx\right\}\left\{\int_{-\infty}^{\infty}\left[\left(\hat{B}^\dagger - \langle B\rangle^*\right)\Psi(x,t)\right]^*\left[\left(\hat{B} - \langle B\rangle\right)\Psi(x,t)\right]dx\right\} \\
&= \left\{\int_{-\infty}^{\infty}\left[\left(\hat{A} - \langle A\rangle\right)\Psi(x,t)\right]^*\left[\left(\hat{A} - \langle A\rangle\right)\Psi(x,t)\right]dx\right\}\left\{\int_{-\infty}^{\infty}\left[\left(\hat{B} - \langle B\rangle\right)\Psi(x,t)\right]^*\left[\left(\hat{B} - \langle B\rangle\right)\Psi(x,t)\right]dx\right\} \\
&= \left[\int_{-\infty}^{\infty}\left|\left(\hat{A} - \langle A\rangle\right)\Psi(x,t)\right|^2 dx\right]\left[\int_{-\infty}^{\infty}\left|\left(\hat{B} - \langle B\rangle\right)\Psi(x,t)\right|^2 dx\right]
\end{aligned}$$

Apply the integral Schwarz inequality (Equation 3.7 on page 93). Note that for a complex number z, $|z|^2 = (\text{Re } z)^2 + (\text{Im } z)^2$ and $\text{Re } z = \frac{1}{2}(z + z^*)$ and $\text{Im } z = \frac{1}{2i}(z - z^*)$.

$$\begin{aligned}
\sigma_A^2\sigma_B^2 &\geq \left|\int_{-\infty}^{\infty}\left[\left(\hat{A} - \langle A\rangle\right)\Psi(x,t)\right]^*\left[\left(\hat{B} - \langle B\rangle\right)\Psi(x,t)\right]dx\right|^2 \\
&= \left\{\text{Re}\int_{-\infty}^{\infty}\left[\left(\hat{A} - \langle A\rangle\right)\Psi(x,t)\right]^*\left[\left(\hat{B} - \langle B\rangle\right)\Psi(x,t)\right]dx\right\}^2 + \left\{\text{Im}\int_{-\infty}^{\infty}\left[\left(\hat{A} - \langle A\rangle\right)\Psi(x,t)\right]^*\left[\left(\hat{B} - \langle B\rangle\right)\Psi(x,t)\right]dx\right\}^2 \\
&= \left\{\frac{1}{2}\int_{-\infty}^{\infty}\left[\left(\hat{A} - \langle A\rangle\right)\Psi(x,t)\right]^*\left[\left(\hat{B} - \langle B\rangle\right)\Psi(x,t)\right]dx + \frac{1}{2}\int_{-\infty}^{\infty}\left[\left(\hat{B} - \langle B\rangle\right)\Psi(x,t)\right]^*\left[\left(\hat{A} - \langle A\rangle\right)\Psi(x,t)\right]dx\right\}^2 \\
&\quad + \left\{\frac{1}{2i}\int_{-\infty}^{\infty}\left[\left(\hat{A} - \langle A\rangle\right)\Psi(x,t)\right]^*\left[\left(\hat{B} - \langle B\rangle\right)\Psi(x,t)\right]dx - \frac{1}{2i}\int_{-\infty}^{\infty}\left[\left(\hat{B} - \langle B\rangle\right)\Psi(x,t)\right]^*\left[\left(\hat{A} - \langle A\rangle\right)\Psi(x,t)\right]dx\right\}^2 \\
&= \left\{\frac{1}{2}\int_{-\infty}^{\infty}\left[\left(\hat{A}^\dagger - \langle A\rangle^*\right)\Psi(x,t)\right]^*\left[\left(\hat{B} - \langle B\rangle\right)\Psi(x,t)\right]dx + \frac{1}{2}\int_{-\infty}^{\infty}\left[\left(\hat{B}^\dagger - \langle B\rangle^*\right)\Psi(x,t)\right]^*\left[\left(\hat{A} - \langle A\rangle\right)\Psi(x,t)\right]dx\right\}^2 \\
&\quad + \left\{\frac{1}{2i}\int_{-\infty}^{\infty}\left[\left(\hat{A}^\dagger - \langle A\rangle^*\right)\Psi(x,t)\right]^*\left[\left(\hat{B} - \langle B\rangle\right)\Psi(x,t)\right]dx - \frac{1}{2i}\int_{-\infty}^{\infty}\left[\left(\hat{B}^\dagger - \langle B\rangle^*\right)\Psi(x,t)\right]^*\left[\left(\hat{A} - \langle A\rangle\right)\Psi(x,t)\right]dx\right\}^2 \\
&= \left\{\frac{1}{2}\int_{-\infty}^{\infty}\left[\left(\hat{A} - \langle A\rangle\right)^\dagger\Psi(x,t)\right]^*\left[\left(\hat{B} - \langle B\rangle\right)\Psi(x,t)\right]dx + \frac{1}{2}\int_{-\infty}^{\infty}\left[\left(\hat{B} - \langle B\rangle\right)^\dagger\Psi(x,t)\right]^*\left[\left(\hat{A} - \langle A\rangle\right)\Psi(x,t)\right]dx\right\}^2 \\
&\quad + \left\{\frac{1}{2i}\int_{-\infty}^{\infty}\left[\left(\hat{A} - \langle A\rangle\right)^\dagger\Psi(x,t)\right]^*\left[\left(\hat{B} - \langle B\rangle\right)\Psi(x,t)\right]dx - \frac{1}{2i}\int_{-\infty}^{\infty}\left[\left(\hat{B} - \langle B\rangle\right)^\dagger\Psi(x,t)\right]^*\left[\left(\hat{A} - \langle A\rangle\right)\Psi(x,t)\right]dx\right\}^2
\end{aligned}$$

Use the fact that $\hat{A} - \langle A \rangle$ and $\hat{B} - \langle B \rangle$ are hermitian operators.

$$\begin{aligned}
\sigma_A^2 \sigma_B^2 \geq & \left[\frac{1}{2}\int_{-\infty}^{\infty} \Psi^*(x,t)\left(\hat{A} - \langle A\rangle\right)\left(\hat{B} - \langle B\rangle\right)\Psi(x,t)\,dx + \frac{1}{2}\int_{-\infty}^{\infty} \Psi^*(x,t)\left(\hat{B} - \langle B\rangle\right)\left(\hat{A} - \langle A\rangle\right)\Psi(x,t)\,dx\right]^2 \\
& + \left[\frac{1}{2i}\int_{-\infty}^{\infty} \Psi^*(x,t)\left(\hat{A} - \langle A\rangle\right)\left(\hat{B} - \langle B\rangle\right)\Psi(x,t)\,dx - \frac{1}{2i}\int_{-\infty}^{\infty} \Psi^*(x,t)\left(\hat{B} - \langle B\rangle\right)\left(\hat{A} - \langle A\rangle\right)\Psi(x,t)\,dx\right]^2 \\
= & \left[\frac{1}{2}\int_{-\infty}^{\infty} \Psi^*(x,t)\left(\hat{A}\hat{B} - \langle B\rangle\hat{A} - \langle A\rangle\hat{B} + \langle A\rangle\langle B\rangle\right)\Psi(x,t)\,dx + \frac{1}{2}\int_{-\infty}^{\infty} \Psi^*(x,t)\left(\hat{B}\hat{A} - \langle A\rangle\hat{B} - \langle B\rangle\hat{A} + \langle B\rangle\langle A\rangle\right)\Psi(x,t)\,dx\right]^2 \\
& + \left[\frac{1}{2i}\int_{-\infty}^{\infty} \Psi^*(x,t)\left(\hat{A}\hat{B} - \langle B\rangle\hat{A} - \langle A\rangle\hat{B} + \langle A\rangle\langle B\rangle\right)\Psi(x,t)\,dx - \frac{1}{2i}\int_{-\infty}^{\infty} \Psi^*(x,t)\left(\hat{B}\hat{A} - \langle A\rangle\hat{B} - \langle B\rangle\hat{A} + \langle B\rangle\langle A\rangle\right)\Psi(x,t)\,dx\right]^2 \\
= & \left[\frac{1}{2}\int_{-\infty}^{\infty} \Psi^*(x,t)\left(\hat{A}\hat{B} - \langle B\rangle\hat{A} - \langle A\rangle\hat{B} + \langle A\rangle\langle B\rangle + \hat{B}\hat{A} - \langle A\rangle\hat{B} - \langle B\rangle\hat{A} + \langle B\rangle\langle A\rangle\right)\Psi(x,t)\,dx\right]^2 \\
& + \left[\frac{1}{2i}\int_{-\infty}^{\infty} \Psi^*(x,t)\left(\hat{A}\hat{B} - \cancel{\langle B\rangle\hat{A}} - \cancel{\langle A\rangle\hat{B}} + \cancel{\langle A\rangle\langle B\rangle} - \hat{B}\hat{A} + \cancel{\langle A\rangle\hat{B}} + \cancel{\langle B\rangle\hat{A}} - \cancel{\langle B\rangle\langle A\rangle}\right)\Psi(x,t)\,dx\right]^2 \\
= & \frac{1}{4}\left[\int_{-\infty}^{\infty} \Psi^*(x,t)\left(\hat{A}\hat{B} + \hat{B}\hat{A} - 2\langle B\rangle\hat{A} - 2\langle A\rangle\hat{B} + 2\langle A\rangle\langle B\rangle\right)\Psi(x,t)\,dx\right]^2 \\
& + \frac{1}{4}\left[-i\int_{-\infty}^{\infty} \Psi^*(x,t)\left(\hat{A}\hat{B} - \hat{B}\hat{A}\right)\Psi(x,t)\,dx\right]^2 \\
= & \frac{1}{4}\left[\int_{-\infty}^{\infty} \Psi^*(x,t)\left(\hat{A}\hat{B} + \hat{B}\hat{A}\right)\Psi(x,t)\,dx - 2\langle B\rangle\int_{-\infty}^{\infty} \Psi^*(x,t)\hat{A}\Psi(x,t)\,dx\right. \\
& \left. -2\langle A\rangle\int_{-\infty}^{\infty} \Psi^*(x,t)\hat{B}\Psi(x,t)\,dx + 2\langle A\rangle\langle B\rangle\int_{-\infty}^{\infty} \Psi^*(x,t)\Psi(x,t)\,dx\right]^2 + \frac{1}{4}\left\{\int_{-\infty}^{\infty} \Psi^*(x,t)\left\{-i\left[\hat{A},\hat{B}\right]\right\}\Psi(x,t)\,dx\right\}^2 \\
= & \frac{1}{4}\left[\int_{-\infty}^{\infty} \Psi^*(x,t)\left(\hat{A}\hat{B} + \hat{B}\hat{A}\right)\Psi(x,t)\,dx - 2\langle B\rangle\langle\Psi\,|\,\hat{A}\,|\,\Psi\rangle - 2\langle A\rangle\langle\Psi\,|\,\hat{B}\,|\,\Psi\rangle + 2\langle A\rangle\langle B\rangle(1)\right]^2 + \frac{1}{4}\left\langle\Psi\,\middle|\,-i\left[\hat{A},\hat{B}\right]\,\middle|\,\Psi\right\rangle^2
\end{aligned}$$

Continue the simplification.

$$\begin{aligned}
\sigma_A^2\sigma_B^2 \geq \frac{1}{4}\left[\int_{-\infty}^{\infty}\Psi^*(x,t)\left(\hat{A}\hat{B}+\hat{B}\hat{A}\right)\Psi(x,t)\,dx - \cancel{2\langle B\rangle\langle A\rangle} - 2\langle A\rangle\langle B\rangle + \cancel{2\langle A\rangle\langle B\rangle}\right]^2 + \frac{1}{4}\left\langle -i\left[\hat{A},\hat{B}\right]\right\rangle^2 \\
&= \frac{1}{4}\left[\int_{-\infty}^{\infty}\Psi^*(x,t)\left(\hat{A}\hat{B}+\hat{B}\hat{A}\right)\Psi(x,t)\,dx - 2\langle A\rangle\langle B\rangle(1)\right]^2 + \frac{1}{4}\left\langle -i\left[\hat{A},\hat{B}\right]\right\rangle^2 \\
&= \frac{1}{4}\left[\int_{-\infty}^{\infty}\Psi^*(x,t)\left(\hat{A}\hat{B}+\hat{B}\hat{A}\right)\Psi(x,t)\,dx - 2\langle A\rangle\langle B\rangle\int_{-\infty}^{\infty}\Psi^*(x,t)\Psi(x,t)\,dx\right]^2 + \frac{1}{4}\left\langle -i\left[\hat{A},\hat{B}\right]\right\rangle^2 \\
&= \frac{1}{4}\left[\int_{-\infty}^{\infty}\Psi^*(x,t)\left(\hat{A}\hat{B}+\hat{B}\hat{A}\right)\Psi(x,t)\,dx + \int_{-\infty}^{\infty}\Psi^*(x,t)\left(-2\langle A\rangle\langle B\rangle\right)\Psi(x,t)\,dx\right]^2 + \frac{1}{4}\left\langle -i\left[\hat{A},\hat{B}\right]\right\rangle^2 \\
&= \frac{1}{4}\left[\int_{-\infty}^{\infty}\Psi^*(x,t)\left(\hat{A}\hat{B}+\hat{B}\hat{A}-2\langle A\rangle\langle B\rangle\right)\Psi(x,t)\,dx\right]^2 + \frac{1}{4}\left\langle -i\left[\hat{A},\hat{B}\right]\right\rangle^2 \\
&= \frac{1}{4}\left\langle \Psi\,\middle|\,\hat{A}\hat{B}+\hat{B}\hat{A}-2\langle A\rangle\langle B\rangle\,\middle|\,\Psi\right\rangle^2 + \frac{1}{4}\left\langle -i\left[\hat{A},\hat{B}\right]\right\rangle^2 \\
&= \frac{1}{4}\left\langle \hat{A}\hat{B}+\hat{B}\hat{A}-2\langle A\rangle\langle B\rangle\right\rangle^2 + \frac{1}{4}\left\langle -i\left[\hat{A},\hat{B}\right]\right\rangle^2 \\
&= \frac{1}{4}\left\{\left\langle -i\left[\hat{A},\hat{B}\right]\right\rangle^2 + \left\langle \hat{A}\hat{B}+\hat{B}\hat{A}-2\langle A\rangle\langle B\rangle\right\rangle^2\right\} \qquad (1)
\end{aligned}$$

Therefore,

$$\sigma_A^2\sigma_B^2 \geq \frac{1}{4}\left(\langle C\rangle^2 + \langle D\rangle^2\right),$$

where

$$\hat{C} = -i\left[\hat{A},\hat{B}\right] \quad \text{and} \quad \hat{D} = \hat{A}\hat{B}+\hat{B}\hat{A}-2\langle A\rangle\langle B\rangle.$$

Part (b)

For the special case that $B = A$, inequality (1) becomes (with Equation 1.12 on page 11)

$$\begin{aligned}
\sigma_A^2\sigma_A^2 &\geq \frac{1}{4}\left\{\left\langle -i\left[\hat{A},\hat{A}\right]\right\rangle^2 + \left\langle \hat{A}\hat{A}+\hat{A}\hat{A}-2\langle A\rangle\langle A\rangle\right\rangle^2\right\} \\
&= \frac{1}{4}\left\{\left\langle -i\left(\hat{A}\hat{A}-\hat{A}\hat{A}\right)\right\rangle^2 + \left\langle 2\hat{A}^2-2\langle A\rangle^2\right\rangle^2\right\} \\
&= \frac{1}{4}\left(0 + 4\left\langle \hat{A}^2-\langle A\rangle^2\right\rangle^2\right) \\
&= \left\langle \hat{A}^2-\langle A\rangle^2\right\rangle^2 \\
&= \left(\langle\hat{A}^2\rangle-\langle A\rangle^2\right)^2 \\
&= \left(\sigma_A^2\right)^2.
\end{aligned}$$